Investigating Chemotaxis in 2D with Moving Finite Elements

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Abstract. We numerically investigate a two dimensional model of aggregation of microglia in two spatial dimensions using the String Gradient Weighted Moving Finite Element method.

Keywords: Chemotaxis, aggregation of microglia, reaction-diffusion, moving finite elements, pattern formation.
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EXTENDED ABSTRACT

Pattern formation, and in particular cell aggregation, is an important phenomenon within the fields of Biology and Chemistry. The application motivating this paper is that of chemotactic cells, known as microglia, in Alzheimer's disease. Several authors have studied chemotactic models analytically as well as using numerical methods. In [1] the authors study a chemotaxis model analytically as well as with a moving mesh method called Moving Mesh Partial Differential Equations (MMPDEs) in 1D. See [2] for a radially symmetric model where the authors also use MMPDEs. Some earlier numerical simulations of pattern formation using a Galerkin Finite Element method in 2D can be found in [3]. Here we propose to investigate numerical simulations in two spatial dimensions of the chemotaxis model from [1]. We propose to compare results to the work in [4], where the authors studied the chemoattraction-chemorepulsion model and parameters from [1] to derive a lower bound for threshold concentration of microglial cells required to coincide with plaque formation. Here we further investigate the corresponding 2D model in [1] for aggregation of microglia in two spatial dimensions using the String Gradient Weighted Moving Finite Element method (SGWMFE).

Moving finite element methods (MFE), and in particular the Gradient Weighted Moving Finite Element methods (GWMFE), are designed for tracking moving shocks and complex structures with a fixed number of mesh nodes. Due to this, these methods are ideal for chemotaxis models where, for the appropriate parameters, the cells aggregate into sharp peaks which need to be resolved. For details of the generalized SGWMFE with applications to the porous medium equation, the shallow water equations and the Gray Scott equations in two spatial dimensions, see [5].

Model equations

We apply the SGWMFE method to the following non-dimensional equations from [1] in two spatial dimensions

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$$\frac{\partial m}{\partial t} = \Delta m - A_1 \nabla \cdot (m \nabla \phi) + A_2 \nabla \cdot (m \nabla \psi),$$
$$\varepsilon_1 \frac{\partial \phi}{\partial t} = \Delta \phi + a^2 (m - \phi), \tag{1}$$
$$\varepsilon_2 \frac{\partial \psi}{\partial t} = \Delta \psi + m - \psi,$$

where the unknown variables m, ϕ and ψ are respectively the cell density and the chemical concentrations of attractant and repellent. The constants $A_1, A_2, \varepsilon_1, \varepsilon_2$ and a are defined in [1]. The equations are defined on a real and bounded domain Ω where the boundary is denoted by Γ. The boundary conditions which hold are zero flux through the boundary Γ. Preliminary results of the solution variables are shown in Figure 1, and the corresponding evolving meshes are shown in Figure 2.

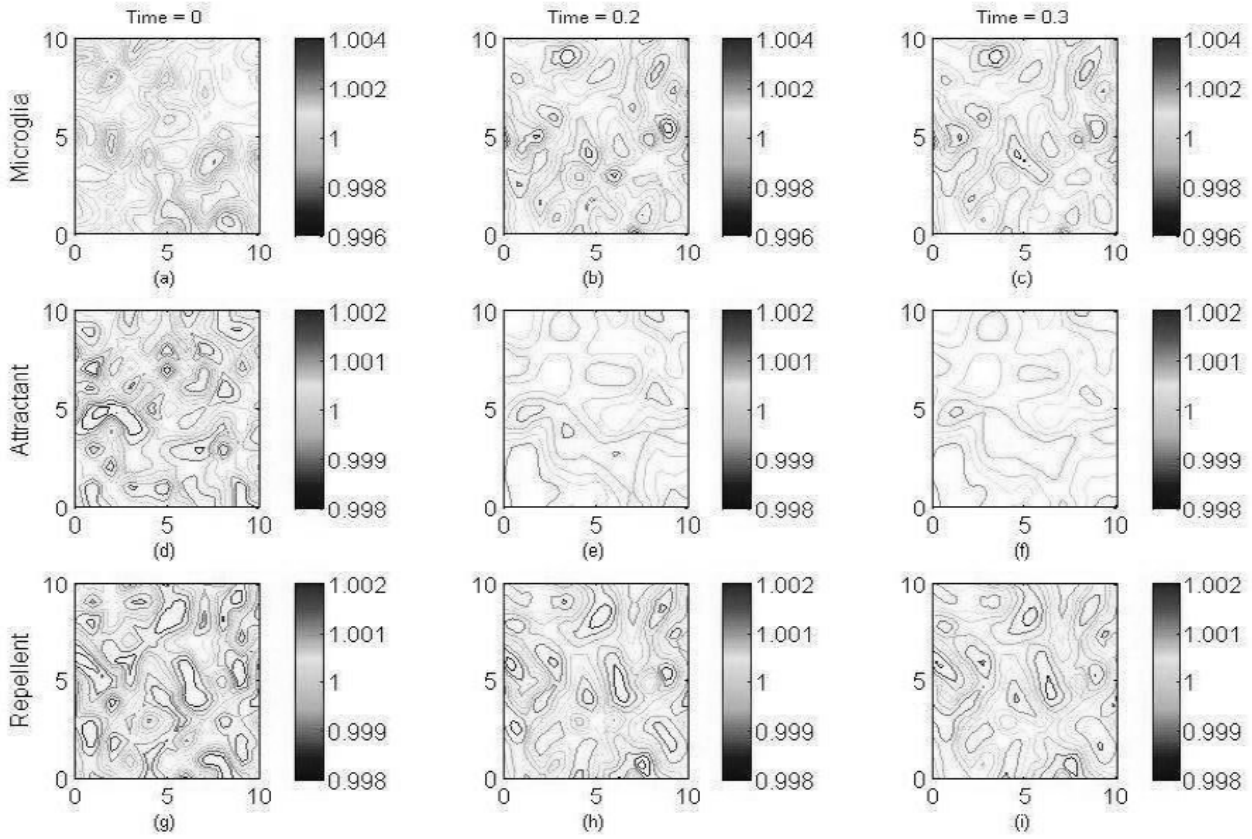

FIGURE 1. Preliminary results using one set of parameter values from [1], with: $A_1 = 2, A_2 = 4, \varepsilon_1 = 2, \varepsilon_2 = 10, a = 2$ and a 10 by 10 physical domain. The equations were solved with SGWMFE using a coarse mesh of 11 by 11 nodes. At time t = 0 the cells and concentrations of attractant and repellent are initialized randomly in the interval (0.998, 1.002).. In this figure (a),(b) and (c) are contour plots of microglia cells, (d),(e) and (f) show the concentration of attractant and (g),(h) and (i) show the concentration of repellent. The first column shows solutions at time t = 0, the second column at time t = 0.2 and the third column at time t = 0.3. Similar to the results shown in [1] for the 1D model, diffusion of the chemorepellent is slower than that of the microglia cells and the chemoattractant. The aggregation is most clear in the contour plots of microglia which show fewer peaks at the later times t = 0.2 and t =0.3 which grow.

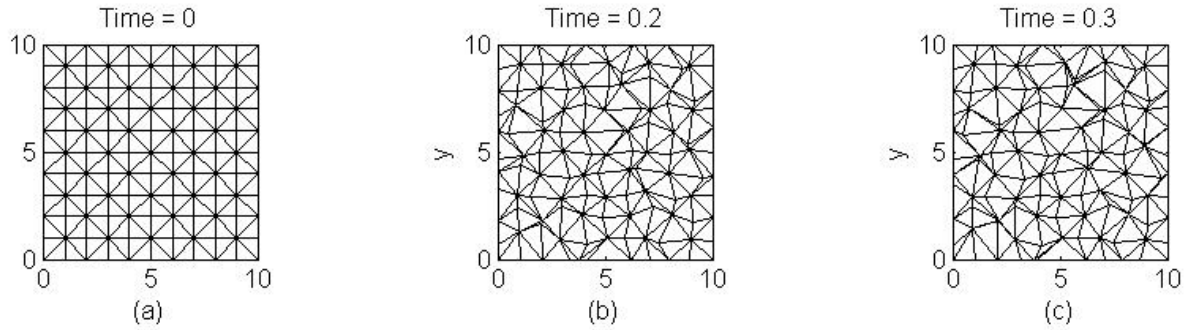

FIGURE 2. Preliminary results using one set of parameter values from [1], with: $A_1 = 2, A_2 = 4, \varepsilon_1 = 2, \varepsilon_2 = 10, a = 2$ and a 10 by 10 physical domain. The equations were solved with SGWMFE using a coarse mesh of 11 by 11 nodes. At time t = 0 the cells and concentrations of attractant and repellent are initialized randomly in the interval (0.998, 1.002). In this figure (a),(b) and (c) are the mesh plots at times t = 0, t = 0.2 and t = 0.3 respectively, corresponding to the sequence of solution plots in Figure 1.

In order to apply SGWMFE to a system of equations such as (1), or other pattern formation equations such as those in [3], we need to extend the theory within the SGWMFE framework to include these terms which have not previously been developed. We have extended the SGWMFE method to include non-linear diffusion terms of different variables such as the last two terms of the first equation in system (1). The authors of [5] develop expressions for the integrals of constant coefficient diffusion and nonlinear diffusion terms within the SGWMFE framework in sections 3.4-3.6 of that paper. We develop the corresponding expressions for the nonlinear diffusion terms of different variables as in equation (1) in the same style as section 3.6 from [5]. This extension results in equations that are analogous to equations (3.29)-(3.32) from [5], as well as analogous expressions for equations (3.20) and (3.22)-(3.25). It is important to note that despite we are considering the nonlinear diffusion terms of different variables in equation (1), the final integral expressions in each case can be written in the form of equation (3.25) from [5], with correspondingly defined coefficients. Though this extension is not shown in detail at this time, this will be included in the full paper.

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REFERENCES

1. M. Luca, A. Chavez-Ross, L. Edelstein-Keshet, A. Mogliner, "Chemotactic Signaling, Microglia, and Alzheimer's Disease Senile Plaques: Is there a Connection?", *Bull. Math. Biol.* **65**, 693-730 (2003).
2. C.J. Budd, R. Carretero-Gonzalez, R.D. Russell., "Precise computations of chemotactic collapse using moving mesh methods", *J. Comp. Phys.* **202**, 463-487 (2005).
3. P. Maini,, M. Myerscough, K. H. Winters, J. Murray, "Bifurcating spatially heterogeneous solutions in a chemotaxis model for biological pattern generation", *Bull. Math. Biol.* **53**, 701-719 (1991).
4. R. A. Quinlan and B. Straughan, "Decay bounds in a model for aggregation of microglia: application to Alzheimer's disease senile plaques", *Proc. R. Soc. A.* **461**, 2887-2897 (2005).
5. A Wacher and I. Sobey, "String Gradient Weighted Moving Finite Elements in Multiple Dimensions with Applications in Two Dimensions", *SIAM J. Sci. Comp.* **29**(2), 459-480 (2007).